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On Dynamic Equilibrium of a Swinging Atwood Machine and Its Stability

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The swinging Atwood machine (SAM) consists of two masses $m_1, m_2 = m_1(1+\varepsilon)$ attached to opposite ends of a massless inextensible thread wound round two massless frictionless pulleys of negligible radius (see [1,2]). The mass m_2 is constrained to move only along a vertical while mass m_1 is allowed to oscillate in a plane and it moves like a pendulum of variable length. Such a system has two degrees of freedom and its Hamiltonian function may be written in the form

$$\mathcal{H} = \frac{p_r^2}{2(2+\varepsilon)} + \frac{p_\varphi^2}{2r^2} + (1+\varepsilon)r - r\cos\varphi, \tag{1}$$

where two variables r, φ describe geometrical configuration of the system, and p_r, p_φ are the corresponding canonically conjugate momenta. Note that equations of motion of the SAM are essentially nonlinear, and their general solution cannot be found in symbolic form. However, there exists a periodic solution which may be represented in the form of power series in a small parameter ε (see [3])

$$\begin{aligned} r(t) &= 1 + \frac{\varepsilon}{16}(1 - 6\cos(2t)) - \frac{3\varepsilon^2}{2048}\left(87 - 92\cos(2t) + 35\cos(4t)\right) + \\ &+ \frac{\varepsilon^3}{131072}\left(4275 - 8166\cos(2t) + 5067\cos(4t) - 1510\cos(6t)\right), \qquad (2) \\ \varphi(t) &= \sqrt{\varepsilon}\left(2\sin t + \frac{53\varepsilon}{192}\sin(3t) + \frac{\varepsilon^2}{81920}\left(14795\sin t - 8495\sin(3t) + 5813\sin(5t)\right)\right). \end{aligned}$$

Note that for $\varepsilon > 0$ the system under consideration has no a static equilibrium state when the coordinates $r(t), \varphi(t)$ are some constants. As periodic solution (2) describes oscillations of the bodies near some equilibrium positions and such a state of the system exists only owing to oscillations one can consider this state as a state of dynamic equilibrium.

As the amplitude of oscillations in (2) is determined by the masses difference ε, it is quite natural to investigate stability of solution (2). Analysing differential equations of the perturbed motion of the SAM and using an infinite determinant method (see [4]), we computed the characteristic exponents in the form of power series

$$\sigma_{1,2} = \pm i, \;\; \sigma_{3,4} = \pm i\frac{\sqrt{3\varepsilon}}{2}\left(1 - \frac{17\varepsilon}{32} + \frac{85\varepsilon^2}{256}\right). \qquad (3)$$

Note that characteristic exponents (3) are different purely imaginary numbers. Thus we prove that periodic solution (2) is stable in linear approximation.

Keywords
Swinging Atwood machine, periodic motion, characteristic exponents, stability

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