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Non-Generic Case of Leap-Frog Algorithm for Optimal Knots Selection in Fitting Reduced Data

Ryszard Kozera[1,2]***, Lyle Noakes***[2] [ryszard_kozera@sggw.edu.pl]

[1] Institute of Information Technology, Warsaw University of Life Sciences - SGGW, Warsaw, Poland
[2] School of Physics, Mathematics and Computing, The University of Western Australia, Perth, Australia

The problem of fitting $n+1$ points $\mathcal{M} = \{x_i\}_{i=0}^{n}$ in arbitrary Euclidean space $\mathbb{E}^m$ is discussed here. The class $\mathcal{I}$ of piecewise C^2 interpolants $\gamma : [0,T] \to \mathbb{E}^m$ satisfying $\gamma(t_i) = x_i$ and $\ddot{\gamma}(t_0) = \ddot{\gamma}(T) = \vec{0}$ admits the *internal knots* $\mathcal{T} = \{t_i\}_{i=1}^{n-1}$ to *vary* satisfying the inequalities $t_0 = 0 < t_1 < ... < t_n = T$. We also stipulate that $\gamma \in \mathcal{I}$ is at least of class C^1 over $\mathcal{T}$ and extends to $C^2([t_i, t_{i+1}])$. Recall that for any fixed set of knots $\mathcal{T}$ the minimization task (over $\mathcal{I}$):

$$\mathcal{J}_T(\gamma) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \|\ddot{\gamma}(t)\|^2 dt \,, \tag{1}$$

yields a unique optimal curve $\gamma_{opt} \in \mathcal{I}$ rendering *a natural cubic spline* γ_{NS} - see [1] or [2]. Consequently, letting the knots $\{t_i\}_{i=1}^{n-1}$ to vary, the task of optimizing (1) over $\mathcal{I}$ reformulates into a search for an optimal natural spline γ_{NS} with knots $\{t_i\}_{i=1}^{n-1}$ relaxed subject to $t_i < t_{i+1}$. By [1], as γ_{NS} is uniquely determined by $\mathcal{T}$, minimizing (1) converts into minimizing a highly non-linear function $J_0(t_1, t_2, \ldots, t_{n-1})$ depending on $n-1$ variables satisfying $t_0 = 0 < t_1 < ... < t_n = T$. Majority of numerical schemes for finding critical points of J_0 lead to numerical difficulties (see e.g. [3]). We discuss here *a Leap-Frog scheme* (see [3] or [4]) which minimizes J_0 based on optimizing a sequence of single variable functions $J_0^{(k)}(t_{k+1})$. The analysis of the sufficient conditions enforcing *unimodality* of $J_0^{(k)}$ for the generic case of Leap-Frog iterations (over each internal sub-interval $[t_i, t_{i+2}]$ with $i = 1, \ldots, n-3$) is addressed in [5]. Our paper extends the latter to the non-generic case of Leap-Frog optimizations taking place over two terminal sub-intervals $[0, t_2]$ and $[t_{n-2}, T]$. Symbolic calculation and numerical tests accompany the theoretical analysis in question.

Keywords
Interpolation, optimization, reduced data

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